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On the Resolution of Equations of the Fifth Degree.

BY EMORY MCCLINTOCK.

Two papers published by Mr. George Paxton Young in the *American Journal of Mathematics* (vol. VI, pp. 65–114) contain discoveries of high importance in the theory of equations. It is a necessary consequence of such publications that others should be attracted to the same field, and I will say at once that what I have now to present is merely in continuation of the work done by Mr. Young.

To relieve the reader from referring repeatedly to Mr. Young's paper on the resolution of equations of the fifth degree, I will first give a brief summary of its main features. The roots $(r_1, r_2, r_3, r_4, r_5)$ of the general equation $x^5+p_2x^3+p_3x^2+p_4x+p_5=0$ being, in accordance with Lagrange's theory, of the form $r_1=u_1+u_2+u_3+u_4$, and so on, where $u_1^5, u_2^5, u_3^5, u_4^5$, are given fractions of the roots of a biquadratic equation, Mr. Young shows, first, that the latter have, in the most general case, this form,

$$\left.\begin{array}{ll} u_1^5=\mathrm{B}+\mathrm{B}'\sqrt{z}+\sqrt{s}, & u_2^5=\mathrm{B}-\mathrm{B}'\sqrt{z}+\sqrt{s_1}, \\ u_4^5=\mathrm{B}+\mathrm{B}'\sqrt{z}-\sqrt{s}, & u_3^5=\mathrm{B}-\mathrm{B}'\sqrt{z}-\sqrt{s_1}, \end{array}\right\} \qquad (1)$$

where $z=1+e^2$, $s=hz+h\sqrt{z}$, $s_1=hz-h\sqrt{z}$, and e, h, B, and B′ are rational; that

$$u_1u_4=g+a\sqrt{z}, \qquad u_2u_3=g-a\sqrt{z}, \qquad (2)$$

where $g=-\frac{1}{10}p_2$, and a is rational; and that

$$\left.\begin{array}{l} u_1^2u_3=k+c\sqrt{z}+(\theta+\phi\sqrt{z})\sqrt{s}, \\ u_4^2u_2=k+c\sqrt{z}-(\theta+\phi\sqrt{z})\sqrt{s}, \\ u_2^2u_1=k-c\sqrt{z}+(\theta-\phi\sqrt{z})\sqrt{s_1}, \\ u_3^2u_4=k-c\sqrt{z}-(\theta-\phi\sqrt{z})\sqrt{s_1}, \end{array}\right\} \qquad (3)$$

where $k=-\frac{1}{20}p_3$, and c, θ, and ϕ are rational. From these and others he derives the six equations following:

$$p_4(g^2-a^2z)=(g^2-a^2z)(5g^2+15a^2z)-20g\,(k^2-c^2z)-20ahez\,(\theta^2-\phi^2z),$$
$$p^5=-4\text{B}+40acz,$$
$$\text{B}''=1,$$
$$\text{B}'''=0,$$
$$hz(\theta^2+\phi^2z+2\theta\phi)=k^2+c^2z-g\,(g^2-a^2z),$$
$$h(\theta^2+\phi^2z+2\theta\phi z)=2kc-a\,(g^2-a^2z).$$

Besides these, a seventh formula is needed, giving B' in terms of the other quantities. The values of B, B', B'', and B''' are indicated, but not expressed. They are, in fact, complicated functions of the other quantities, and would each contain, if written at length, from twenty to thirty terms. Mr. Young now calls attention to the general fact that it is always possible to eliminate five of the unknown quantities from his six equations, leaving a single equation of some degree (presumably of a very high degree, except in cases solvable algebraically), from which the numerical value of the remaining unknown quantity can be ascertained. He has therefore succeeded in showing the correct form of the roots of the general quintic, and in proving that it is not impossible to determine and exhibit the roots in that form. He has not, however, reduced the method to a manageable process, except for a single special case of unusual simplicity ($a=0$). To supply this deficiency, and to show in detail how the roots may be determined in their normal form, without transformation, by the aid of a resolvent sextic, is the first object of the present paper. In doing this it will be shown that Mr. Young's original equations may be simplified, and two of the four unwieldy formulæ dispensed with. For each of the trinomials

$$x^5+p_4x+p_5=0,\quad x^5+p_3x^2+p_5=0,$$

there will be afforded a choice between two sextics, one a special case of what I call the first resolvent, the other of the second. After discussing these resolvents, I shall close by making some observations on certain classes of solvable quintics.

ON THE GENERAL QUINTIC.

"By the resolution of a quintic, I mean the expression of its roots in terms of those of its resolvent sextic."* Tried by this standard, Mr. Young's work needs to be continued and completed. He has shown the normal form of the roots fully and satisfactorily, and has proved the possibility of exhibiting them without any other than a linear transformation of the original equation. But

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*Cockle, "Resolution of Quintics," *Quarterly Journal*, IV, 5.

though he has shown that such exhibition is possible, by means of the numerical solution of the six equations quoted, it remains a mere theoretical possibility. The elimination of five of the six unknown quantities, if attempted by what might be called main force, would in most cases involve very great labor.

The methods of resolution devised by Lagrange and Vandermonde were too intricate to be practicable. A method of resolving the trinomial $x^5 + 10\delta x^2 + \zeta = 0$ was given in 1860 by Mr. (now Sir James) Cockle, in the paper from which a sentence was quoted in the foregoing paragraph; but it did not give the full form of the root, nor did it apply to the general quintic. Another process was gradually evolved by the labors of Messrs. Cockle and Harley, during a period of perhaps fifteen years, in numerous papers appearing in various journals. This process reached its culmination in a memoir presented by Professor Cayley to the Royal Society on Feb. 20, 1861. Mr. Cockle showed that any quintic for which the "resolvent product" is evanescent is solvable by radicals, and that the general quintic can, by two successive transformations, aided by his resolvent sextic, be brought into that solvable form. The resolvent product (θ) would be represented in Mr. Young's notation by $625 u_1 u_2 u_3 u_4$. Mr. Cockle's sextic determined the numerical value of θ for the trinomial $x^5 + 10\delta x^2 + \zeta = 0$, and enabled him to transform any equation from this form to a solvable form, the trinomial itself being first obtained by a Tschirnhausen transformation. That the second transformation was not essential was shown by Mr. Harley, who employed a modification of Mr. Cockle's sextic (previously made by the latter) in which the unknown quantity is t (say $5a\sqrt{z}$), and proved that the roots of the trinomial are rational functions of t. Merely for the trinomial, t is not more simple than θ; for the general quintic, however, it is much more simple, being, in Mr. Young's notation, $5a\sqrt{z}$ against $625(g^2 - a^2 z)$. Recognizing this great difference, Mr. Cayley calculated for the general quintic the sextic in t, or rather in $10\sqrt{5a\sqrt{z}}$, showing that its roots are rational functions of those of the quintic, and vice versa.* In his own words, "The roots of the given quintic are each of them rational functions of the roots of the auxiliary equation, so that the theory of the solution of an equation of the fifth order appears to have been carried to its extreme limit." The sextic so discovered by him is said, according to Mr. Harley, "to take an extremely simple form; it may, in fact, be regarded as canonical in the theory of equations of the fifth degree."

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*I gather this from the brief abstract in the *Proceedings*, not having as yet seen the memoir. See Cockle, "Researches in the Higher Algebra," *Manchester Memoirs*, XV, 131; Harley, "On the Theory of Quintics," *Quarterly Journal*, III, 343.

On the one hand, therefore, we have the auxiliary sextic thus evolved, whose roots are known to be functions of those of the given quintic. On the other hand Mr. Young now shows the form of the roots of the general quintic, with six simultaneous equations from which to extract their values. It only remains to connect these two theories. The steps which I shall indicate are not necessarily the best for this purpose, and I shall be glad if any one improves upon them; but they have at least this merit, that at no point is it necessary to solve an equation of any degree above the first. They therefore make it possible, if any one will take the trouble, to express the roots directly and visibly in terms of those of a resolvent sextic.

As regards notation, I shall follow that of Mr. Young, except where departure seems clearly warranted by the nature of the results. In lieu of his expression for the general equation, $x^5 - 10gx^3 - 20kx^2 + p_4 x + p_5 = 0$, useful to him as somewhat simplifying his formulæ, I shall employ the normal form

$$x^5 + 10\gamma x^3 + 10\delta x^2 + 5\varepsilon x + \zeta = 0.$$

The reader will kindly notice the respective values of the symbols employed for the coefficients. As thus changed, formulæ (2) become

$$u_1 u_4 = -\gamma + az^{\frac{1}{2}}, \quad u_2 u_3 = -\gamma - az^{\frac{1}{2}}. \tag{4}$$

Observing these and (1), and putting $v = a^2 z$, we see that

$$\begin{aligned} u_1^5 u_4^5 &= z^{\frac{1}{2}} a(v^2 + 10v\gamma^2 + 5\gamma^4) - (5\gamma v^2 + 10\gamma^3 v + \gamma^5) \\ &= z^{\frac{1}{2}}\mathrm{F} + \mathrm{G}, \text{ suppose,} \\ &= [\mathrm{B} + \mathrm{B}' z^{\frac{1}{2}} + (hz + hz^{\frac{1}{2}})^{\frac{1}{2}}][\mathrm{B} + \mathrm{B}' z^{\frac{1}{2}} - (hz + hz^{\frac{1}{2}})^{\frac{1}{2}}] \\ &= z^{\frac{1}{2}}(2\mathrm{BB}' - h) + \mathrm{B}^2 + \mathrm{B}'^2 z - hz. \end{aligned}$$

Everything being rational except $z^{\frac{1}{2}}$, we have

$$\mathrm{F} = 2\mathrm{BB}' - h, \tag{5}$$

$$\mathrm{G} = \mathrm{B}^2 + \mathrm{B}'^2 z - hz. \tag{6}$$

From these

$$\left.\begin{aligned} \mathrm{B}' &= \mathrm{B} + \left(\frac{\mathrm{B}^2 e^2 - \mathrm{F}z + \mathrm{G}}{z}\right)^{\frac{1}{2}}, \\ h &= 2\mathrm{BB}' - \mathrm{F}. \end{aligned}\right\} \tag{7}$$

This determines h and B' when v, a, and B are known, since $z = va^{-2}$, and these three quantities are all we need to seek in order to assign the elements of the root as in (1). We thus avoid the long and complex expressions for B' and B'', and shall see that we have also obviated the necessity of determining the values of the quantities θ and ϕ.

Five of Mr. Young's six equations will still be useful to us. These are, after making the changes of notation already indicated, and putting $c = \frac{1}{2}at$, and (merely for shortness) $\gamma^2 - v = \mathrm{V}$, $\gamma^2 + v = \mathrm{V}'$,

$$\mathrm{V}\varepsilon = \mathrm{V}\gamma^2 + 3\mathrm{V}v + \gamma\delta^2 - \gamma vt^2 - 4heva^{-1}(\theta^2 - \phi^2 z), \tag{8}$$

$$\zeta = 20vt - 4\mathrm{B}, \tag{9}$$

$$\mathrm{B}''' = 0, \tag{10}$$

$$4hz(\theta^2 + \phi^2 z + 2\theta\phi) = \delta^2 + vt^2 + 4\mathrm{V}\gamma, \tag{11}$$

$$ha^{-1}(\theta^2 + \phi^2 z + 2\theta\phi z) = -\tfrac{1}{2}\delta t - \mathrm{V}. \tag{12}$$

Calculating B and B''' as indicated by him, but shortening the work by employing the letters V and V' as above stated, and also by putting $\mathrm{Q} = \frac{1}{2}\mathrm{V}'(vt^2 + \delta^2) + 2\gamma\delta vt + 2\gamma\mathrm{V}v + \gamma\mathrm{V}\mathrm{V}'$, and $\mathrm{R} = \gamma vt^2 + \mathrm{V}'\delta t + 2\gamma^2\mathrm{V} + \mathrm{V}\mathrm{V}' + \gamma\delta^2$, we have

$$\mathrm{V}^2\mathrm{B} = heva^{-1}(\theta^2 - \phi^2 z)(\mathrm{V}'t + 2\gamma\delta) - \tfrac{1}{2}\delta\mathrm{Q} + \tfrac{1}{2}vt\mathrm{R}, \tag{13}$$

$$e\mathrm{V}^2\mathrm{B}''' = 0 = \theta\left[\gamma(vt^2 - \delta^2)ae + \mathrm{Q} + a\mathrm{R}\right] - \phi\left[\tfrac{1}{2}\mathrm{V}'(vt^2 - \delta^2)e - \mathrm{Q} - va^{-1}\mathrm{R}\right]. \tag{14}$$

What we have to do is to extract in some way from these equations the values of B, v, and a, in terms of the known quantities γ, δ, ε, and ζ; and as B is given by (9) when v and t are known, we may say that the values desired are those of v, t, and a.

Let $\mathrm{P} = \mathrm{V}\gamma^2 + 3\mathrm{V}v + \gamma\delta^2 - \mathrm{V}\varepsilon - \gamma vt^2$. Then, from (8),

$$4heva^{-1}(\theta^2 - \phi^2 z) = \mathrm{P}. \tag{15}$$

Eliminating B from (9) and (13),

$$4heva^{-1}(\theta^2 - \phi^2 z)(\mathrm{V}'t + 2\gamma\delta) = 20\mathrm{V}^2 vt - \mathrm{V}^2\zeta + 2\delta\mathrm{Q} - 2vt\mathrm{R}. \tag{16}$$

Dividing (16) by (15), and clearing the result of fractions, we have this equation in v and t:

$$\mathrm{V}'t\mathrm{P} + 2\gamma\delta\mathrm{P} = 20\mathrm{V}^2 vt - \mathrm{V}^2\zeta + 2\delta\mathrm{Q} - 2vt\mathrm{R}. \tag{17}$$

Another such equation is needed. Put $\mathrm{M} = \delta^2 + vt^2 + 4\mathrm{V}\gamma$, and $\mathrm{N} = -2\delta vt - 4\mathrm{V}v$. We take the following from (11) and (12) respectively:

$$4heva^{-1}(\theta^2 + \phi^2 z + 2\theta\phi) = \mathrm{M}ea, \tag{18}$$

$$4heva^{-1}(\theta^2 + \phi^2 z + 2\theta\phi z) = \mathrm{N}e. \tag{19}$$

Adding (15) respectively to (18) and (19), and dividing the first result by the second,

$$\frac{\theta + \varphi}{\theta + \varphi z} = \frac{\mathrm{M}ea + \mathrm{P}}{\mathrm{N}e + \mathrm{P}}; \tag{20}$$

subtracting instead of adding,

$$\frac{\theta + \varphi z}{\theta z + \varphi z} = \frac{\mathrm{M}ea - \mathrm{P}}{\mathrm{N}e - \mathrm{P}}. \tag{21}$$

Multiplying these equations together,

$$z^{-1}(\mathrm{N}^2e^2 - \mathrm{P}^2) = \mathrm{M}^2e^2a^2 - \mathrm{P}^2;$$

whence, since $z = 1 + e^2$, and $a^2 z = v$,

$$\mathrm{N}^2 = \mathrm{M}^2 v - \mathrm{P}^2, \tag{22}$$

or

$$(2\delta v t + 4\mathrm{V}v)^2 = (\delta^2 + vt^2 + 4\gamma\mathrm{V})^2 v - \mathrm{P}^2, \tag{23}$$

a second equation in v and t.

We have now, in (17) and (23), two equations in v and t. In any case not critical we can, by division or some other usual process, depress these so as to have an equation of the first degree in v or t, for use in case one of them is known; and we can eliminate either v or t, thus obtaining a resolvent sextic, as will be seen further on. With numerical coefficients, no difficulty will occur, nor is it necessary to frame a formula for t in terms of v. The problem may be considered solved, therefore, as regards v and t, and it only remains to determine a. For this we need an equation of the first degree, not containing θ, ϕ, h, z, or e.

From (14), (20), and (21), the following equations are readily obtained, remembering that $v = a^2 z$ and $z = 1 + e^2$:

$$\frac{\theta}{\varphi} = \frac{\mathrm{T}e - \mathrm{Q} - \mathrm{R}va^{-1}}{\mathrm{W}ae + \mathrm{Q} + \mathrm{R}a} = \frac{\mathrm{M}va^{-1} + \mathrm{P}c - \mathrm{N}}{\mathrm{N} - \mathrm{M}a} \tag{24}$$

$$= \frac{\mathrm{M}v - \mathrm{N}va^{-1}}{\mathrm{N}a + \mathrm{P}ae - \mathrm{M}v}. \tag{25}$$

Here $\mathrm{T} = \frac{1}{2}\mathrm{V}'(vt^2 - \delta^2)$, and $\mathrm{W} = \gamma(vt^2 - \delta^2)$. When (24) and (25) are cleared of fractions, respectively, and $va^{-1} - a$ is replaced by ae^2, wherever that combination is observed, we obtain from them these:

$$aep = am - n,$$

$$aeq = an - mv.$$

Here $m = \mathrm{NW} - \mathrm{PR} - \mathrm{MT}$, $n = \mathrm{MW}v + \mathrm{PQ} - \mathrm{NT}$, $p = \mathrm{MQ} + \mathrm{NR} + \mathrm{PW}$, and $q = \mathrm{MR}v + \mathrm{NQ} + \mathrm{PT}$. Eliminating ae, finally,

$$a = \frac{pmv - qn}{pn - qm}. \tag{26}$$

For determining the roots of the general quintic, therefore, the following steps are to be pursued:

First, ascertain v or t by eliminating t or v from (17) and (23), or by employing one of the resolvent sextics known to be producible by such elimination, and then obtain the other of these two quantities by means of the same equations;

Secondly, determine a by (26) and z by $a^2 z = v$;

Finally, ascertain B, h, and B' by (9) and (7), thus obtaining all that is necessary to fill out the root-forms (1). We have still to discuss the possible resolvent sextics.

On the Trinomial $x^5+5\varepsilon x+\zeta=0$.

When $\gamma=\delta=0$, our formulæ become much simplified. Here $\mathrm{V}'=-\mathrm{V}=v$, $\mathrm{Q}=\frac{1}{2}v^2t^2$, $\mathrm{R}=-v^2$, $\mathrm{P}=v\varepsilon-3v^2$, and from (17) and (23), after striking out common factors, and putting $u=25v$, we have

$$tu-t\varepsilon-\zeta=0, \tag{27}$$

$$u^2-ut^4-6\varepsilon u+25\varepsilon^2=0. \tag{28}$$

Eliminating u, we have this simple sextic in t:

$$\varepsilon t^6+\zeta t^5-20\varepsilon^2t^2+4\varepsilon\zeta t-\zeta^2=0. \tag{29}$$

Eliminating t, on the other hand, we have this simple sextic in u:

$$(u-\varepsilon)^4(u^2-6\varepsilon u+25\varepsilon^2)-u\zeta^4=0. \tag{30}$$

When by either of these we ascertain the value of u or t, the value of the other is given by (27), and that of v by $u=25v$. To illustrate these sextics, we may take the equation $x^5+\frac{625}{4}x+3750=0$, whose elements are given in detail by Mr. Young. Here $\varepsilon=\frac{125}{4}$, and $\zeta=3750$. Then by (29) $t=5$, or by (30) $u=\frac{3125}{4}$; and these values satisfy (27). The following are other forms of (30):

$$(u^3-5\varepsilon u^2+15\varepsilon^2u+5\varepsilon^3)^2=u(\zeta^4+256\varepsilon^5), \tag{31}$$

$$u^6-10\varepsilon u^5+55\varepsilon^2u^4-140\varepsilon^3u^3+175\varepsilon^4u^2-(\zeta^4+106\varepsilon^5)u+25\varepsilon^6=0. \tag{32}$$

If we put $u=\varepsilon y$, equation (30) takes this form:

$$(y-1)^4(y^2-6y+25)=y\varepsilon^{-5}\zeta^4. \tag{33}$$

In this compact equation we have the unknown quantity expressed directly as a function of the parameter $\varepsilon^{-5}\zeta^4$. Were tables for such purposes worth calculating, a table of single entry might be constructed with great ease, showing a value of the parameter for each value of y, and therefore the converse, which would supply all that is needed to exhibit the roots of the trinomial in their normal form.

These various results are doubtless new, except that (31) may be readily derived from Mr. Cayley's sextic. If we take the square roots of both sides of (31), the result is

$$u^3-5\varepsilon u^2+15\varepsilon^2u\mp\sqrt{(\zeta^4+256\varepsilon^5)}\sqrt{u}+5\varepsilon^3=0, \tag{34}$$

twin sextics for finding $\pm\sqrt{u}$. If in Mr. Cayley's sextic (I have not seen it, but the fact cannot be otherwise) we were to put $\alpha = 1$, $\beta = \gamma = \delta = 0$, we should reproduce (34), except that the unknown quantity ($\sqrt{u}$) would be replaced by another equal to $\sqrt{u} \times 2\sqrt{5}$. Equation (31) is, in fact, a special case of a general resolvent in u which, as we shall see hereafter, may be immediately derived from Mr. Cayley's sextic. Similarly, equation (29) is a special case of a resolvent in t.

For determining a by means of (26), we find that, when $\gamma = \delta = 0$, $\mathrm{M} = vt^2$, $\mathrm{N} = 4v^2$, $\mathrm{T} = \frac{1}{2}v^2t^2$, and $\mathrm{W} = 0$; so that, omitting the factor v^3, $m = \varepsilon - 3v - \frac{1}{2}t^4$, $n = \frac{1}{2}\varepsilon t^2 - \frac{7}{2}vt^2$, $p = \frac{1}{2}t^4 - 4v$, and $q = \frac{1}{2}\varepsilon t^2 - \frac{1}{2}vt^2$. For determining h and B' by means of (7), we have $\mathrm{F} = av^2$, $\mathrm{G} = 0$.

On the Trinomial $x^5 + 10\delta x^2 + \zeta = 0$.

When $\gamma = \varepsilon = 0$, we have $\mathrm{V}' = -\mathrm{V} = v$, $\mathrm{Q} = \frac{1}{2}v^2t^2 + \frac{1}{2}v\delta^2$, $\mathrm{R} = v\delta t - v^2$, $\mathrm{P} = -3v^2$, and putting $u = 25v$ as before, we readily obtain from (17) and (23) the following:

$$25\delta^3 - \delta ut^2 = u\zeta - u^2t, \tag{35}$$

$$(25\delta^2 - ut^2)^2 = u^3 - 16\delta u^2t. \tag{36}$$

Squaring (35), multiplying (36) by δ^2, and comparing the two results, we find that

$$u^2t^2 - 2\zeta ut - \delta^2u + 16\delta^3t + \zeta^2 = 0. \tag{37}$$

We have in (35) and (37) two quadratic equations in t, and also in u. If we wish to eliminate x from the two equations $ax^2 + bx + c = 0$, $dx^2 + ex + f = 0$, we find that $b(bd - ae)(cd - af) - a(cd - af)^2 - c(bd - ae)^2 = 0$. Eliminating u and t respectively in this manner from (35) and (37), we have at once these two sextics:

$$16\delta^3t^6 + \zeta^2t^5 - 50\delta^2\zeta t^4 + 40\delta^4t^3 + 15\delta\zeta^2t^2 + 59\delta^3\zeta t - \zeta^3 + 25\delta^5 = 0, \tag{38}$$

$$u^6 - 20\delta\zeta u^4 - 160\delta^4u^3 + 100\delta^2\zeta^2u^2 - (1856\delta^5\zeta + \zeta^4)u + 6400\delta^8 = 0. \tag{39}$$

Here (38) is a special case of the second resolvent, that in t before spoken of, and is new; (39) is a special case of the first resolvent (that in u), immediately to be discussed, and is not new, being the well-known sextic of Sir James Cockle. This equation has had an interesting history. For years the focus of acute researches, when this trinomial was considered as affording the best path to the solution of the quintic, it was at length set aside, and its place was taken by Mr. Cayley's sextic, or rather by that special case of the latter which belongs to this trinomial. It now appears again as one of the most direct aids in the resolution of the equation to which it refers. As announced by Mr. Cockle, our u was represented by $-\frac{1}{25}\theta$, δ by $-\frac{1}{2}\mathrm{Q}$, and ζ by E. He also exhibited it in another form, to which this corresponds:

$$(u^3 - 10\delta\zeta u - 80\,\delta^4)^2 = (3456\delta^5\zeta + \zeta^4)u. \tag{40}$$

By taking the square roots of both sides of this equation, he produced another which had as its unknown quantity what corresponded to $5\sqrt{u}$, and which was afterwards in substance generalized by Mr. Cayley as already stated.

On the First Resolvent.

I have repeatedly referred to two resolvent sextics, the numerical solution of one of which gives the value of u, leading directly, under Mr. Young's theory as herein extended, to the determination of all the roots of the general quintic in their normal and complete form. This first resolvent, in what seems to be its most convenient shape, is as follows:

$$\begin{aligned}25\,[&\tfrac{1}{5}u^3 - (3\gamma^2 + \varepsilon)u^2 + (15\gamma^4 - 2\gamma^2\varepsilon + 8\gamma\delta^2 - 2\delta\zeta + 3\varepsilon^2)u - 25\gamma^6 + 35\gamma^4\varepsilon \\ &\qquad - 40\gamma^3\delta^2 - 2\gamma^2\delta\zeta - 11\gamma^2\varepsilon^2 + \gamma\zeta^2 + 28\gamma\delta^2\varepsilon - 2\delta\varepsilon\zeta - 16\delta^4 + \varepsilon^3]^2 \\ = 8u\,[&432\gamma^5\zeta^2 - 1440\gamma^4\delta\varepsilon\zeta + 800\gamma^4\varepsilon^3 + 640\gamma^3\delta^3\zeta - 400\gamma^3\delta^2\varepsilon^2 - 180\gamma^3\varepsilon\zeta^2 \\ &+ 330\gamma^2\delta^2\zeta^2 + 560\gamma^2\delta\varepsilon^2\zeta - 320\gamma^2\varepsilon^4 - 1260\gamma\delta^3\varepsilon\zeta + 720\gamma\delta^2\varepsilon^3 - 15\gamma\delta\zeta^3 \\ &+ 20\gamma\varepsilon^2\zeta^2 + 432\delta^5\zeta - 270\delta^4\varepsilon^2 + 45\delta^2\varepsilon\zeta^2 - 80\delta\varepsilon^3\zeta + 32\varepsilon^5 + \tfrac{1}{8}\zeta^4].\end{aligned} \tag{41}$$

From this equation, when $\gamma = \delta = 0$, we have (31) as a special case, and when $\gamma = \varepsilon = 0$, (40).

That such a sextic may exist, as a result of eliminating t from (17) and (23), is evident upon observing the special cases just mentioned. The literal part of the coefficients might be written out with ease, having regard to the necessary weight of each, but the determination from those equations of the numerical part, by elimination or otherwise, would be a work of much labor. It is unnecessary, as we have seen, since the unknown quantity (ϕ) in Mr. Cayley's sextic is equal to $2\sqrt{(5u)}$. To obtain our first resolvent, therefore, we have only to substitute the latter value in that sextic in lieu of ϕ, to put, with Mr. Young, $\alpha = 1$ and $\beta = 0$, these being the coefficients of x^5 and x^4, and, bringing the irrational terms to one side of the equation, to square both sides. We are, in fact, reversing for the general equation the step taken for the trinomial when Mr. Cockle's θ was replaced by a multiple of its square root, though in a direction somewhat different. While θ differs, as regards the trinomial, from u only by a numerical factor, it is otherwise for the general equation; since $u = 25a^2z$, while $\theta = 625\,(\gamma^2 - a^2z)$, the two differing only by a factor when $\gamma = 0$, as happens in the trinomial.

Mr. Cayley's sextic involves a cyclic function of the roots, namely,

$$\phi = r_1r_2 + r_2r_3 + r_3r_4 + r_4r_5 + r_5r_1 - r_2r_4 - r_4r_1 - r_1r_3 - r_3r_5 - r_5r_2.$$

To demonstrate our first resolvent more directly than by employing that sextic would require the building up of a sextic having for its roots $\frac{1}{20}\phi^2$ and the five similar cyclic functions of the roots of the quintic. There is no reason for this being done, unless it should happen, what is hardly probable, that the coefficients of u could be determined more easily than those of ϕ. This question naturally arises: should we not dispense with the symbol u, and with the modified resolvent (41), and rely wholly on ϕ and Mr. Cayley's sextic? Several reasons appear conclusive against this course.

1. Mr. Young's whole theory is based on the employment of symbols having rational values only. Now u is rational, and ϕ is not; $u = 25a^2z$, $\phi = 10\sqrt{5a}\sqrt{z}$.

2. The connection of the resolvent with, and its derivation from, the two equations in v and t would be obscured by the introduction of the symbol ϕ, and the same might be said of the whole theory.

3. As $u = \frac{1}{20}\phi^2$, the use of ϕ would add needless numerical complication to all the processes.

4. Since the value of ϕ is only needed in the form of ϕ^2, it would be a roundabout proceeding to determine first the irrational quantity ϕ by the original sextic, and then to square it, when u can be found directly by our resolvent (41).

5. One of the coefficients in the original sextic is irrational; those of (41) are all rational.

6. The coefficient of ϕ in the original sextic is, as Mr. Cayley states, the square root of the discriminant of the quintic, multiplied by a numerical factor (irrational); the coefficient of u in (41) is that discriminant itself, unchanged.

The values of the coefficients in (41) were taken by me, with the necessary changes, from a translation in the *Analyst* (1877) of a pamphlet published in 1861 by Mr. Adolf von der Schulenburg,* which contained a sextic for $\frac{1}{2}\phi$, which I presume to be identical with that given early in the same year by Mr. Cayley. The coefficient of u has been verified by Dr. Salmon's expression for the discriminant.

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* *Auflösung der Gleichungen Fünften Grades*, translated by Professor A. B. Nelson. The formulæ given for the roots are erroneous.

ON THE SECOND RESOLVENT.

When $\gamma = 0$, the resolvent in t, otherwise complex and as yet uninvestigated, appears as a sextic of remarkable properties, and decidedly preferable in all respects to that in u. In this case, when we eliminate v from (17) and (23), we find that $v\mathrm{A} = -\mathrm{C}$, where A is the quintic in $-t$, and C its canonizant, namely, $\mathrm{A} = -t^5 + 10\delta t^2 - 5\varepsilon t + \zeta$, and $\mathrm{C} = \delta^2 t^3 - \delta\varepsilon t^2 + (\varepsilon^2 - \delta\zeta)t - \delta^3$. This simple relation gives the value of v when that of t has been obtained by means of the second resolvent; the terms of which, ascertained by the same process of elimination, are

$$\delta^3\mathrm{A}^2 + (\zeta + \varepsilon t + \delta t^2)\,\mathrm{AC} + 25t\mathrm{C}^2 = 0, \tag{42}$$

or, in full,

$$\begin{aligned}&(16\delta^4 + 2\delta\varepsilon\zeta - \varepsilon^3)t^6 + (\delta\zeta^2 - 44\delta^3\varepsilon - \varepsilon^2\zeta)t^5 + (75\delta^2\varepsilon^2 - 50\delta^3\zeta)t^4 \\ &+ (40\delta^5 - 40\delta\varepsilon^3 + 30\delta^2\varepsilon\zeta)t^3 + (15\delta^2\zeta^2 - 55\delta^4\varepsilon - 30\delta\varepsilon^2\zeta + 20\varepsilon^4)t^2 \\ &+ (3\delta\varepsilon\zeta^2 - 20\delta^3\varepsilon^2 + 59\delta^4\zeta - 4\varepsilon^3\zeta)t + 25\delta^6 - 6\delta^3\varepsilon\zeta - \delta\zeta^3 + \varepsilon^2\zeta^2 = 0.\end{aligned}$$

From this, when $\delta = 0$ and $\varepsilon = 0$ respectively, we have (29) and (38) as special cases.

It is obvious that these two resolvents do not comprise all that are possible. If any one of the unknown quantities becomes known, or any relation between them, the quintic becomes solvable, as Mr. Young has pointed out. Hence any auxiliary equation which determines for us any one of the unknown quantities, or any known function of more than one, will serve as a resolvent.

Mr. Young's theory might have been stated differently by him; as, for example, by making his elements u_1, u_2, etc. five times as large. This would make changes necessary throughout, but would not necessarily cause any change in the coefficients of the resolvents.

It will be seen, by reference to (17) and (23), that a resolvent might be found in which the unknown quantity would have the value of vt. Judging from the trinomials, its expression must be complex.

ON CERTAIN SOLVABLE QUINTICS.

Mr. Young has given a method for determining the roots in the critical case $a = 0$; that is to say, in the case $u = 0$, since $u = 25a^2z$. He has not, I think, indicated any criterion by which such quintics can be recognized when met with. We have in the first resolvent the needed criterion: when the terms unaffected by u vanish, $u = 0$. Quoting those terms, we have $u = 0$ when

$$25\gamma^6 - 35\gamma^4\varepsilon + 40\gamma^3\delta^2 + 2\gamma^2\delta\zeta + 11\gamma^2\varepsilon^2 - \gamma\zeta^2 - 28\gamma\delta^2\varepsilon - 2\delta\varepsilon\zeta + 16\delta^4 - \varepsilon^3 = 0. \tag{43}$$

For examples of this important class of solvable quintics, it is sufficient to refer the reader to Mr. Young's paper.

Sir James Cockle is the discoverer of a criterion covering another large class of solvable quintics, that namely in which his θ (or $25\gamma^2 - u$) vanishes. The criterion so discovered was afterwards recalculated and verified by Mr. Harley, and may be found in his paper on Symmetric Products, in vol. XV of the *Manchester Memoirs.* We can now reproduce its equivalent by putting $u = 25\gamma^2$ in the first resolvent, with this result ($\mathrm{D} =$ discriminant):

$$\begin{aligned}[1600\gamma^6 - 640\gamma^4\varepsilon + 160\gamma^3\delta^2 - 52\gamma^2\delta\zeta + 64\gamma^2\varepsilon^2 + \gamma\zeta^2 \\ + 28\gamma\delta^2\varepsilon - 2\delta\varepsilon\zeta - 16\delta^4 + \varepsilon^3]^2 = \gamma^2\mathrm{D}.\end{aligned} \tag{44}$$

The form of this criterion is very different from that given by Mr. Cockle, but the two must be identical. When $\varepsilon = 4\gamma^2$, and $\delta = 0$, the criterion is satisfied, the quintic in question being well-known as the solvable form of De Moivre; when $\gamma = 0$, (44) gives $-2\delta\varepsilon\zeta - 16\delta^4 + \varepsilon^3 = 0$, equally well known as the solvable form of Euler. The latter, by the way, is also a special case of (43). In the class of cases now under consideration, one or more of the elements u_1, u_2, u_3, u_4, is non-existent. We have here, as in the case $u = 0$, a critical value of u, and our equations need to be replaced in some points by others. The methods of solution given by Mr Harley in the paper referred to seem well adapted to this class of cases.

The condition of equal roots is the vanishing of the discriminant, and it has long been known that the solution by radicals of such quintics is possible. Perhaps we have here as simple a method of solving them as any that has been perfected. The second member of the resolvent (41) is $\mathrm{D}u$. When D vanishes, the quintic has equal roots, and we may obtain the value of u, and therefore, by the method now presented, the values of the roots, from the cubic equation to which (41) is reduced, namely,

$$\begin{aligned}\tfrac{1}{5}u^3 - (3\gamma^2 + \varepsilon)u^2 + (15\gamma^4 - 2\gamma^2\varepsilon + 8\gamma\delta^2 - 2\delta\zeta + 3\varepsilon^2)u - 25\gamma^6 + 35\gamma^4\varepsilon - 40\gamma^3\delta^2 \\ - 2\gamma^2\delta\zeta - 11\gamma^2\varepsilon^2 + \gamma\zeta^2 + 28\gamma\delta^2\varepsilon - 2\delta\varepsilon\zeta - 16\delta^4 + \varepsilon^3 = 0.\end{aligned} \tag{45}$$

No doubt an important class of solvable quintics will be found to appear when $t = 0$, for which the absolute term of the second resolvent will, when placed equal to zero, afford the necessary criterion. If, for example, we have a trinomial $x^5 + 10\delta^2 + \zeta = 0$, where $\zeta^3 = 25\delta^5$, we know that it is a solvable form.

Other criteria for solvable quintics might be derived, if they were worth having, in numbers without limit, by assigning to u and t, in the two resolvents, different values in terms of the coefficients.

It is conceivably questionable whether the use in any way of knowledge derived from a resolvent sextic does not vitiate the strictly algebraic solution, or "solution by radicals" of a quintic. If, as I have no doubt, it is permissible, it becomes something of a question in casuistry how far we should go in multiplying solvable cases. An infinitely large accumulation of solvable quintics would leave none of the other sort, by which to illustrate the fact that the quintic cannot be solved algebraically; yet it would not change the fact.

PROCEDURE IN CRITICAL CASES.

For two of the classes of solvable quintics just considered, the general course of procedure sketched in this paper requires modification, owing to the appearance of vanishing fractions or vanishing equations.

When $v=0$, we know that $a=0$, since $v=a^2z=a^2(1+e^2)$, where e is rational; also, since $at=2c$, $vt=0$, and $vt^2=4c^2z$. Equation (23) shows at once that $\mathrm{P}=0$, whence

$$4c^2z=\gamma^3+\delta^2-\gamma\varepsilon. \tag{46}$$

To get the value of z, we need to learn that of c. For this purpose we may modify (26) as follows. Let $\mathrm{M}=\frac{1}{2}\mathrm{M}'$, $\mathrm{N}=t^{-1}\mathrm{N}'$, $\mathrm{P}=t^{-1}\mathrm{P}'$, $\mathrm{Q}=\gamma^2\mathrm{Q}'$, $\mathrm{R}=\gamma^2\mathrm{R}'ca^{-1}$, $\mathrm{T}=\gamma^2\mathrm{T}'$, so that, in this case, $\mathrm{M}'=2\delta^2+8c^2z+8\gamma^3$, $\mathrm{N}'=-8\delta c^2z$, $\mathrm{P}'=\delta^3+2\delta\gamma^3-\gamma^2\zeta+4\delta c^2z$, $\mathrm{Q}'=2c^2z+\frac{1}{2}\delta^2+\gamma^3$, $\mathrm{R}'=2\delta$, and $\mathrm{T}'=2c^2z-\frac{1}{2}\delta^2$. Then (24) and (25) take this shape:

$$\frac{\theta}{\varphi}=\frac{\mathrm{T}'e-\mathrm{Q}'-\mathrm{R}'cz}{\mathrm{Q}'+\mathrm{R}'c}=\frac{\mathrm{M}'cz+\mathrm{P}'e-\mathrm{N}'}{\mathrm{N}'-\mathrm{M}'c} \tag{47}$$

$$=\frac{\mathrm{M}'cz-\mathrm{N}'z}{\mathrm{N}'+\mathrm{P}'e-\mathrm{M}'cz}\,. \tag{48}$$

When (47) and (48) are cleared of fractions, respectively, and $z-1$ is replaced by e^2, we have

$$cep'=cm'-n'$$

$$ceq'=cn'-m'c^2z,$$

where $m'=-\mathrm{P}'\mathrm{R}'-\mathrm{M}'\mathrm{T}'$, $n'=\mathrm{P}'\mathrm{Q}'-\mathrm{N}'\mathrm{T}'$, $p'=\mathrm{M}'\mathrm{Q}'+\mathrm{N}'\mathrm{R}'$, and $q'=\mathrm{M}'\mathrm{R}'c^2z+\mathrm{N}'\mathrm{Q}'+\mathrm{P}'\mathrm{T}'$; and eliminating ce,

$$c=\frac{p'm'c^2z-q'n'}{p'n'-q'm'}\,. \tag{49}$$

Hence, when $v=a=0$, we must first determine c^2z by (46), then c and z by (49), then B by (9), which gives $\mathrm{B}=-\frac{1}{4}\zeta$, and B' and h by (7), where in this case $\mathrm{F}=0$ and $\mathrm{G}=-\gamma^5$, so that

$$\mathrm{B}'=\mathrm{B}+\left(\frac{\mathrm{B}^2e^2-\gamma^5}{z}\right)^{\frac{1}{2}}, \tag{50}$$

$$h=2\mathrm{BB}'. \tag{51}$$

In the special case $v=a=c=0$, it is readily shown that $\sqrt{z}=\frac{\varepsilon^2+\gamma^4}{\varepsilon^2-\gamma^4}\cdot$

When $v=\gamma^2$, it would no doubt be possible to perfect a method of procedure to take the place of that given by Mr. Harley, but, so far as I can now see, it would not be advantageous. In any given case of this sort, when desired, the roots can be shown in Mr. Young's form. Thus, for example, if $z=1$, $t=0$, and $a=-\gamma$, we have DeMoivre's case already cited. Here $\text{B}=-\frac{1}{4}\zeta$, by (9); and we readily find by (7), since $\text{F}z=\text{G}=-16\gamma^5$, and $e=0$, that $\text{B}'=\text{B}$, and $h=\frac{1}{8}\zeta^2+16\gamma^5$.

MILWAUKEE, March 7, 1884.

POSTSCRIPT, *March* 24.—A different form for the roots, probably new,* may be obtained very readily. Let $l_1=u_1u_4$, and $l_2=u_2u_3$; also let $m_1+n_1=u_1^2u_3$, $m_1-n_1=u_4^2u_2$, $m_2+n_2=u_2^2u_1$, and $m_2-n_2=u_3^2u_4$. Then $m_1^2-n_1^2=l_1^2l_2$, and $m_2^2-n_2^2=l_2^2l_1$, and since $u_1^5=(u_1^2u_3)^2(u_2^2u_1)(u_2u_3)^{-2}$, and similarly for u_2^5 and the rest, we have at once

$$\left.\begin{array}{ll} u_1^5=(m_1+n_1)^2(m_2+n_2)l_2^{-2}, & u_2^5=(m_2+n_2)^2(m_1-n_1)l_1^{-2}, \\ u_3^5=(m_2-n_2)^2(m_1+n_1)l_1^{-2}, & u_4^5=(m_1-n_1)^2(m_2-n_2)l_2^{-2}. \end{array}\right\} \qquad (52)$$

Here, following our notation, it will be seen that $l_1=-\gamma+\sqrt{v}$, $l_2=-\gamma-\sqrt{v}$, $2m_1=-\delta+t\sqrt{v}$, $2m_2=-\delta-t\sqrt{v}$, and, on determining the proper signs for the radicals by reference to a special case, that $n_1=-\sqrt{(m_1^2-l_1^2l_2)}$, and $n_2=\sqrt{(m_2^2-l_2^2l_1)}$. Thus when v and t are ascertained, we have the roots of the general equation expressible in terms of these quantities, or of one of them; in this way fully satisfying Sir James Cockle's definition, already quoted, of the problem of the Resolution of the Quintic.

I have hitherto left the fundamental equations (17) and (23) unexpanded, believing them in that form better adapted for use in numerical calculations. I find, however, that they both contain the factor $v-\gamma^2$, and that when this factor is expelled they may be written in full as follows, each being given in two forms, arranged respectively according to the powers of v and t:

$$\left.\begin{array}{r} 25tv^2-(\gamma t^3+\delta t^2+\varepsilon t+10\gamma^2t+\zeta)v+\gamma\text{H}t+\text{K}=0, \\ \gamma vt^3+\delta vt^2-(25v^2-10\gamma^2v-\varepsilon v+\gamma\text{H})t+\zeta v-\text{K}=0; \end{array}\right\} \qquad (17_a)$$

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* The German writer Schulenburg, already referred to, would have reached this form in substance had he not made an error in one of his equations, which was perpetuated in those which followed. Any critic may have noticed this, and so found the true form. But his whole view of the subject was different from the present, nor did he even attempt to show how to determine the quantities corresponding to our t.

$$\left.\begin{aligned} &25v^3-(t^4-14\gamma t^2+16\delta t+35\gamma^2+6\varepsilon)v^2+(2\mathrm{H}t^2+2\gamma\delta^2\\ &\qquad\qquad+4\gamma^2\varepsilon+11\gamma^4+\varepsilon^2)v-\mathrm{H}^2=0,\\ &v^2t^4-(14\gamma v^2+2\mathrm{H}v)t^2+16\delta v^2t-25v^3+35\gamma^2v^2+6\varepsilon v^2\\ &\qquad\qquad-2\gamma\delta^2v-4\gamma^2\varepsilon v-11\gamma^4v-\varepsilon^2v+\mathrm{H}^2=0.\end{aligned}\right\} \qquad (23_a)$$

Here $\mathrm{H}=\gamma^3-\gamma\varepsilon+\delta^2$, and $\mathrm{K}=\gamma^2\zeta-2\gamma\delta\varepsilon+\delta^3$. By subtracting $(23_a)\times t$ from $(17_a)\times v$, there results this second quadratic in v:

$$\begin{aligned}&(t^5-15\gamma t^3+15\delta t^2+25\gamma^2t+5\varepsilon t-\zeta)v^2\\ &\qquad-(2\mathrm{H}t^3+\gamma\delta^2t+5\gamma^2\varepsilon t+10\gamma^4t+\varepsilon^2t-\mathrm{K})v+\mathrm{H}^2t=0.\end{aligned} \qquad (53)$$

If we represent (17_a) by $av^2+bv+c=0$, and (53) by $dv^2+ev+f=0$, we find on depressing them that $v=\dfrac{cd-af}{ae-bd}=\dfrac{bf-ce}{cd-af}$, whence

$$(cd-af)^2+(bd-ae)(bf-ce)=0. \qquad (54)$$

Dividing this, after expansion, by $-\gamma^3t^4-\gamma^2\delta t^3+(16\gamma^4+\gamma\delta^2-2\gamma^2\varepsilon)t^2+(\delta^3-2\gamma\delta\varepsilon)t$, we obtain at once the second resolvent, represented by (42) when $\gamma=0$. This important sextic proves to be of the simplest possible nature, being in short

$$5^2\mathrm{C}^2-\mathrm{AL}=0, \qquad (55)$$

where A is the quintic in $-t$, namely, $-t^5-10\gamma t^3+10\delta t^2-5\varepsilon t+\zeta$, L its simplest linear covariant, and C its canonizant. That is to say, $\mathrm{L}=l_0+l_1t$, and $\mathrm{C}=c_0+c_1t+c_2t^2+c_3t^3$, where $l_0=9\gamma^4\zeta-20\gamma^3\delta\varepsilon+10\gamma^2\delta^3+8\gamma^2\varepsilon\zeta-12\gamma\delta\varepsilon^2-2\gamma\delta^2\zeta+6\delta^3\varepsilon+\delta\zeta^2-\varepsilon^2\zeta$, $l_1=15\gamma^4\varepsilon-10\gamma^3\delta^2+2\gamma^2\delta\zeta-14\gamma^2\varepsilon^2+22\gamma\delta^2\varepsilon-\gamma\zeta^2-9\delta^4+2\delta\varepsilon\zeta-\varepsilon^3$, $c_0=2\gamma\delta\varepsilon-\gamma^2\zeta-\delta^3$, $c_1=\gamma\delta^2-\gamma^2\varepsilon-\delta\zeta+\varepsilon^2$, $c_2=-\gamma^2\delta+\gamma\zeta-\delta\varepsilon$, and $c_3=\gamma^3-\gamma\varepsilon+\delta^2$. After these quantities are calculated, the resolvent (55) may be used in this expanded form,

$$\begin{aligned}&(l_1+25c_3^2)t^6+(l_0+50c_2c_3)t^5+5(2\gamma l_1+5c_2^2+10c_1c_3)t^4+10(\gamma l_0-\delta l_1+5c_0c_3+5c_1c_2)t^3\\ &\qquad+5(-2\delta l_0+\varepsilon l_1+5c_1^2+10c_0c_2)t^2+(5\varepsilon l_0-\zeta l_1+50c_0c_1)t-\zeta l_0+25c_0^2=0.\end{aligned} \qquad (56)$$

I have followed Mr. Young in treating the quintic without its second term, but it will be seen that neither of the resolvents is thus limited in its nature.

Note.—On page 314 of the present volume of this *Journal* I gave formulae (No. 52) for the elements of the roots of the general equation of the fifth degree. In practice, these formulae give the roots of two conjugate equations, according to the sign taken for $\sqrt{v}$. A change in the sign causes an interchange between l_1 and l_2, m_1 and m_2, n_1 and n_2. If the formulae fail at first to give the root of any given equation, it is only necessary to make these interchanges, or (the same thing in final effect) to change the sign of n_1 or of n_2.

When v and t have been determined for any quintic, we can at once construct the conjugate quintic. Equation (23_a) is a quadratic in ε, one value being the given value of ε, the other being that of ε in the conjugate quintic. From the latter value, that of ζ may be obtained by (17_a). For example, if the given quintic be $x^5+\frac{625}{4}x+3750=0$, the conjugate is $x^5+\frac{3125}{4}x+3125=0$, and a root of the latter is given by (52) after making the interchanges referred to, or after changing the sign of n_1 or of n_2.

April 5, 1884.